1. Researchers propose two models for estimating the weight of accumulated flammable material in a forest, where weight is measured in appropriate units and t is the number of years since the most recent bush fire. The models, and their graphs, are:

$$W_1 = 10(1 - e^{-0.1t})$$
$$W_2 = 2\sqrt{t}$$

(a) The models produce very similar predictions for $t \leq 20$. Explain mathematically why the predictions differ substantially for larger values of t.

(2 marks)

(b) Use **one step** of Newton's method to estimate a value of t at which both models predict the same weight of material. (Hint: let $f(t) = W_1 - W_2$, so you need to solve $f(t) = 0$. Use an initial estimate of $t = 9$ years, and note that $W_1' = e^{-0.1t}$ and $W_2' = \dfrac{1}{\sqrt{t}}$.)

(5 marks)

2. A 'traditional' method of estimating the weight of a pig is:

- Measure its heart girth (circumference) x, in inches
- Measure its length y, in inches
- The weight p, in pounds, is approximately $p = \dfrac{x^2 y}{400}$.

(a) Demonstrate how to make the formula for p dimensionally homogeneous. (3 marks)

(b) Let W be the weight of the pig expressed in kg. Write an expression for the weight p (in pounds) in terms of W. Include units in your answers. (Hint: there are 2.2 pounds per kg weight.)

(2 marks)

(c) Let G and L be the girth and length of the pig (respectively) in metres. Write an expression for the girth x (in inches) in terms of G, and an expression for the length y (in inches) in terms of L. Include units in your answers. (Hint: there are 39.4 inches per metre.)

(2 marks)

(d) Recall that the weight p in pounds is $p = \dfrac{x^2 y}{400}$, where girth x and length y are in inches.

Use your answers to Parts (b) and (c) to show that when the girth G and length L are measured in metres, the weight W expressed in kg is as follows (include units in your answer):

$$W \approx 69.5 G^2 L.$$

(3 marks)

(e) Assume instead that a pig is modelled as a cylinder of circumference G metres and length L metres. Show that its estimated weight W expressed in kg is as follows (include units in your answer):

$$W \approx 79.6 G^2 L.$$

(Hint: there is 1000 kg weight per m^3 of pig. The circumference of a circle of radius r is $2\pi r$. The volume of a cylinder of radius r and length l is $\pi r^2 l$.)

(4 marks)

(f) In Part (e), the weight of the pig expressed in kg is modelled as $W \approx 79.6G^2L$.

(i) On the left-hand set of axes, sketch (and label) two graphs showing W versus L for:
- pigs with a **fixed girth**, say G_1; and
- pigs with a **fixed** girth, equal to $2G_1$.

(ii) On the right-hand set of axes, sketch (and label) two graphs showing W versus G for:
- pigs with a **fixed length**, say L_1; and
- pigs with a **fixed length**, equal to $2L_1$.

(4 marks)

(iii) Consider two pigs, P_1 and P_2, with identical estimated weights. If the radius of P_1 is 20% larger than the radius of P_2, find the ratio of the length of P_1 to that of P_2.

(3 marks)

3. Three measures of the effectiveness of a test for a given condition are the *sensitivity*, *specificity* and *accuracy* of the test. These are defined as follows, where A, B, C and D are the values in the table, and N is the total population.

- Sensitivity $= \dfrac{A}{A+C}$
- Specificity $= \dfrac{D}{B+D}$
- Accuracy $= \dfrac{A+D}{N}$

Condition: Is the condition present?

		Yes	No
Test	Positive	A	B
	Negative	C	D

(a) (i) A test for a certain type of cancer displays the following characteristics when applied to 100 individuals:

	Cancer present	Cancer absent
Test positive	7	6
Test negative	3	?

Find the value missing from the table, and the accuracy, sensitivity and specificity of the test.

(3 marks)

Missing Value: ______ Accuracy: ______ Sensitivity: ______ Specificity: ______

(ii) List two potential harms arising from a false negative test result for cancer (represented by C in the table), and two potential harms arising from a false positive test result (B in the table).

(2 marks)

(b) It is known that 10% of the members of a male population have prostate cancer. A blood test for the cancer has a sensitivity of 34.9% and a specificity of 63.1%.

(i) Complete the following table. (Write any working outside the table.)

(4 marks)

Condition: Does the man have prostate cancer?

		Yes	No
Blood test	Positive		
	Negative		

(ii) Find the probability that a man who tests positive actually has prostate cancer.

(1 mark)

(iii) Find the probability that a man who tests negative actually has prostate cancer.

(1 mark)

(iv) On the basis of your answers to Parts (ii) and (iii), what advice would you give to a man who was considering taking the blood test?

(1 mark)

(c) Consider a group of animals (birds and mammals) in which:

* 60% of the animals are birds, and 40% are mammals;
* 50% of birds are female and 50% of mammals are female;
* all female birds lay eggs and 25% of female mammals lay eggs;
* no male birds or mammals lay eggs.

Scientists propose two tests to determine whether (or not) **a given animal lays eggs**. In each case, complete the table when the test is applied to 100 individuals, and find the accuracy, sensitivity and specificity of the test. (Recall that accuracy, sensitivity and specificity are defined above Part (a).)

(i) To identify whether an animal lays eggs, the proposed test is: "Is it a bird?".

(4 marks)

Condition: Does the animal lay eggs?

		Yes	No
Proposed test: Is the animal a bird?	Positive		
	Negative		

Accuracy = ________ Sensitivity = ________ Specificity = ________

(ii) To identify whether an animal lays eggs, the proposed test is: "Is it female?".

(4 marks)

Proposed test: Is the animal female?

	Condition: Does the animal lay eggs? Yes	No
Positive		
Negative		

Accuracy = ________ Sensitivity = ________ Specificity = ________

4. In lectures, we used the following function $d(t)$ to model the probability that a woman will die of breast cancer **prior to reaching age t years**, where t is between 30 and 85:

$$d(t) = \frac{1}{43} \times \frac{1}{55^2} \times (t - 30)^2 .$$

Note that $d(t)$ simplifies (approximately) to: $D(t) = 7.7 \times 10^{-6} \times (t - 30)^2$.

(a) Give one or more physical reasons why $D(t)$ is always:

(i) increasing as t gets larger. (1 mark)

(ii) increasing at an increasing rate as t gets larger. (1 mark)

(b) Use $D(t)$ to estimate the probability that a young woman will die of breast cancer **before reaching** age 60 years.

(1 mark)

(c) The derivative of $D(t)$, written $D'(t)$, is $D'(t) = 7.7 \times 10^{-6} \times (2t - 60)$. Explain briefly why $D'(t)$ gives the probability that a woman will die of breast cancer **at** age t.

(2 marks)

(d) Find the expected number of **annual** breast cancer deaths in a group of 1000 women all aged 60. (Hint: refer to Part (c).)

(2 marks)

(e) When a person dies at an age younger than the population-wide life expectancy, the number of *Years of Potential Life Lost* (YPLL) equals the difference between the life expectancy and their actual life span. Why is YPLL a useful concept, and what role could it play in public health policy?

(3 marks)

(f) The life expectancy of Australian women is 85 years, so the YPLL for a woman who dies at age t years, $t \leq 85$, is $y(t) = (85 - t)$. Estimate the **expected total YPLL due to breast cancer deaths in a year** for a group of 1000 women all aged 60. (Hint: refer to Part (d).)

(2 marks)

(g) Briefly justify why the expected total YPLL due to breast cancer deaths in a year for 1000 women of age t years is $L(t) = 7.7 \times 10^{-3} \times (85 - t) \times (2t - 60)$.

(3 marks)

(h) In Queensland, all women aged from 40 to 69 are entitled to free breast cancer screening. Justify this, with particular reference to $L(t)$ (defined in Part (g)). The graph of $L(t)$ is:

(3 marks)

(i) Let T be the time at which $L'(T) = 0$, where L' is the derivative of L. Find the (approximate) value of T using the graph in Part (h). Briefly interpret the physical meaning of your answer, and explain how it is relevant to society.

(4 marks)

5. (a) A household generates electricity using a solar panel. The rate of power generation at any time is shown on the graph, for times between $t = 0$ (sunrise) and $t = 12$ (sunset).

Write a function $G(t)$ for the rate of power generation at any time t between 0 hours and 12 hours. Note that $G(t)$ is a sin function, with exactly one half of a cycle occurring between $t = 0$ and $t = 12$.

(3 marks)

(b) Find B, where $B = \displaystyle\int_0^{12} G(t)\, dt$.

(Hint: $\displaystyle\int a \sin bt\, dt = -\frac{a}{b}\cos bt + C$. Also, $\cos 0 = 1, \cos\pi = -1, \cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = 0$.)

(5 marks)

(c) The household also consumes power. The following graph shows the rate of power consumption at any time t, with some values shown in the table. (Note that at time $t = 8$, there is a sudden increase in power consumption from 800 W to 1000 W; this is denoted "800; 1000" in the table.)

Time (hrs)	0	2	8	10	12
Power (watts)	1000	800	800; 1000	1000	1400

Let C represent the area under the power consumption curve. Find the **exact** value of C, and include units in your answer.

(7 marks)

(d) The following partial Python program is intended to find the area C under the power consumption curve in Part (c). **In the box**, write all of the output. (Write rough working outside the box.)

```
t = array([0, 2, 8, 10, 12])
p = array([1000, 800, 800, 1000, 1400])
tA = 0
a = zeros(5)
i = 0
while i<4:
    a[i] = p[i] * (t[i+1] - t[i])
    tA = tA + a[i]
    print "a[i] = ",a[i]
    i = i + 1
print "AUC = ", tA, "units. "
```

(7 marks)

The output is:

(e) The Python code does not produce the correct value for the area under the curve. Briefly explain the error(s) in the program.

(3 marks)

6. In a population at high risk of contracting cancer, each individual is classified as: Undiagnosed (U), Positive (P), Treatment (T) or Deceased (D). There is a test for the cancer, but there are false positive and false negative test results. Each month:

- Each Undiagnosed person will either test positive and move to the Positive category (10% likelihood), return a false negative test result and Die (3%) or return a true negative test result (87%) and remain Undiagnosed.
- Each Positive person will move to the Treatment category (70% likelihood), or move to the Undiagnosed category (30%) because their earlier test result was a false positive.
- Each person in the Treatment category will either recover and move to the Undiagnosed category (40% likelihood), require treatment for another month (40%), or Die (20%).

Researchers wish to develop a model based on a system of differential equations.

(a) On the following diagram, mark all possible transitions between the four categories, including the probability of each transition.

(7 marks)

(b) Let $U(t)$, $P(t)$, $T(t)$ and $D(t)$ be the number of people in each category at any time t. Equations for U' and D' are:

$$\begin{aligned} U' &= -0.13U + 0.3P + 0.4T \\ D' &= 0.2T + 0.03U \end{aligned}$$

Write equations for P' and T'.

(2 marks)

(c) From Part (b), equations for U' and D' are:

$$\begin{aligned} U' &= -0.13U + 0.3P + 0.4T \\ D' &= 0.2T + 0.03U \end{aligned}$$

From these equations, show that stable solutions can only occur if $U = P = T = 0$. (Hint: the number of people in any category cannot be negative.)

(3 marks)

(d) Find the value(s) of D at stable solutions to the equations, and explain your answer briefly.

(2 marks)

(e) In a particular population, at time $t = 0$, the functions and derivatives have the following values:

Function values	$U(0) = 800$	$P(0) = 100$	$T(0) = 50$	$D(0) = 0$
Derivative values	$U'(0) = -54$	$P'(0) = -20$	$T'(0) = 40$	$D'(0) = 34$

Apply **two** steps of Euler's method with a **step size of one month** to estimate the number of people in each category at time $t = 2$ months.

(7 marks)

(f) Euler's method was applied to these equations for a period of 12 months, producing the following graphs. (Note that the initial population in this question is different from that in Part (e).)

The four curves on the graph correspond to $D(t)$, $P(t)$, $T(t)$ and $U(t)$ in some order. Complete the following, and and justify your answers carefully.

(4 marks)

$D(t)$ corresponds to Graph number: ____ $P(t)$ corresponds to Graph number: ____

$T(t)$ corresponds to Graph number: ____ $U(t)$ corresponds to Graph number: ____

Reasons: