INTRODUCTION TO GRAPHS WHICH ARE LINKED STRUCTURES

Roger B. Eggleton
Mathematics Department, Illinois State University
Normal, IL 61790, U.S.A.

For Severino V. Gervacio, on his 65th birthday.

Abstract: This paper provides an introduction to graphs which are linked pairs, linked chains, or linked cycles. These structures allow for a generalized notion of graph factorization, permitting regular decomposition of many graphs into isomorphic links.

1. An unconventional "factorization"

Consider the graph H in Fig. 1. This is an identity graph, that is, its only automorphism is the identity map on its vertices. Nevertheless it has a more subtle structural regularity which is of interest.

For any subset V of the vertex set of H, let $H[V]$ be the subgraph of H induced by V. In particular, for $A = \{1,2,3,4,5,6,7\}$ and $B = \{1,2,3,4,8,9,10\}$ let $G_0 = H[A]$ and $G_1 = H[B]$. These induced subgraphs both have order 7 and size 6. Indeed, each is isomorphic to the smallest identity tree T (a 3-legged spider with legs of length 1, 2 and 3), so H contains two isomorphic induced subgraphs. In fact $G_0 \cup G_1 = H$, so these two subgraphs comprise H. Thus H "factorizes" into two copies of the tree T. However $G_0 \cap G_1 = H[1,2,3,4] \cong 2P_2$, so G_0 and G_1 are not edge-disjoint. This "factorization" differs from a conventional factorization in two ways: it is less demanding insofar as its "factors" may share edges, but it is more demanding insofar as its "factors" are not simply subgraphs, they are subgraphs which are induced by their vertex sets.

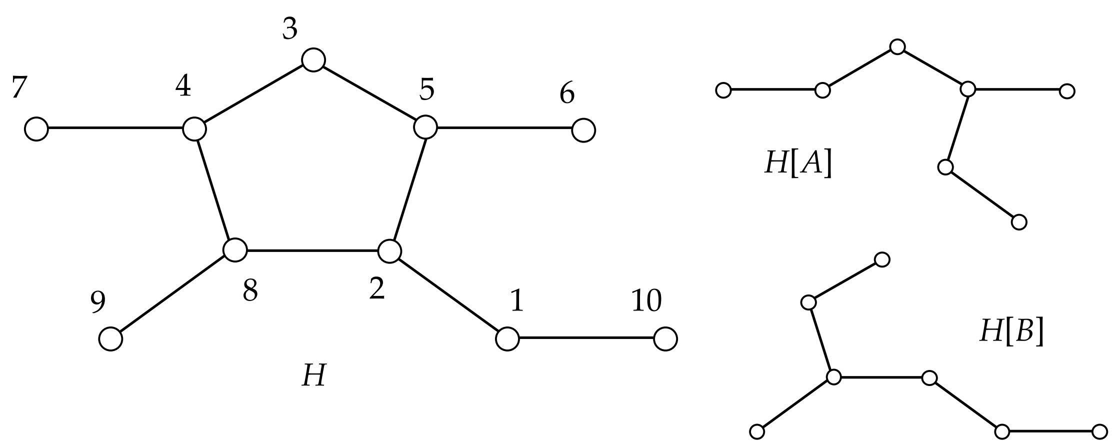

Figure 1. An unconventional "factorization" of H.

2. Linked pairs

Let us generalize from the example in Fig. 1. Choose any finite simple graph G of order $g \geq 2$, and an induced proper subgraph K of order k, where $1 \leq k < g$. Label the vertices of K with the positive integers $[1..k]$, and extend this labeling so that the positive integers $[1..g]$ label all the vertices of G. Let K^* be an induced subgraph of G isomorphic to K; since K^* could be K itself, there is always at least one available choice for K^*. Let $\sigma: K^* \to K$ be an isomorphism. Extend σ to a bijection $\tau:[1..g] \to [1..k] \cup [(g+1)..(2g-k)]$, by requiring $\tau(i) = \sigma(i)$ for every vertex i of K^*, and $\tau(j) = g+i$ if j is the ith vertex of G not in K^*. Then τ induces a graph isomorphism from G to G', a copy of G labeled by $[1..k] \cup [(g+1)..(2g-k)]$. The resulting graph, $H = G \cup G'$, is labeled by the positive integers $[1..(2g-k)]$. Note that if $A = [1..g]$ and $B = [1..k] \cup [(g+1)..(2g-k)]$, then $H = G_0 \cup G_1$, where $G_0 = G = H[A]$ and $G_1 = G' = H[B]$, so $K = H[A \cap B] = H[1..k]$.

The new graph H is the *linked pair* with *initial link* $G_0 = G$, *copy link* $G_1 = G'$, and *link isomorphism* τ. The subgraph $K = G_0 \cap G_1$ is the *kernel* of the linked pair, K^* is the *prekernel*, and σ is the *shift*. The procedure is the *linked pair construction* with *ingredients* G, K, K^* and σ. These notions were introduced and developed in collaboration with Peter Adams and James MacDougall, in [**1**]. The purpose of the present paper is to give a less formal introduction to the main constructions in that paper.

For example, let G be T, the 3-legged spider with legs of length 1, 2 and 3, labeled as in Fig. 2. Let K be the induced subgraph $2P_2$ shown, and let $K^* = K$. There are eight possible isomorphisms from K^* to K. Choose one of them to be σ, as specified in Fig. 2. One possible relabeling of G with [1..7] so that K is labeled with [1..4] yields G_0 in Fig. 2. Then τ is the bijection $\tau(1) = 3$, $\tau(2) = 4$, $\tau(3) = 2$, $\tau(4) = 1$, $\tau(5) = 8$, $\tau(6) = 9$, $\tau(7) = 10$, so $\tau(G_0) = G_1$ in Fig. 2. The resulting linked pair $H = G_0 \cup G_1$ is precisely the graph H in Fig. 1.

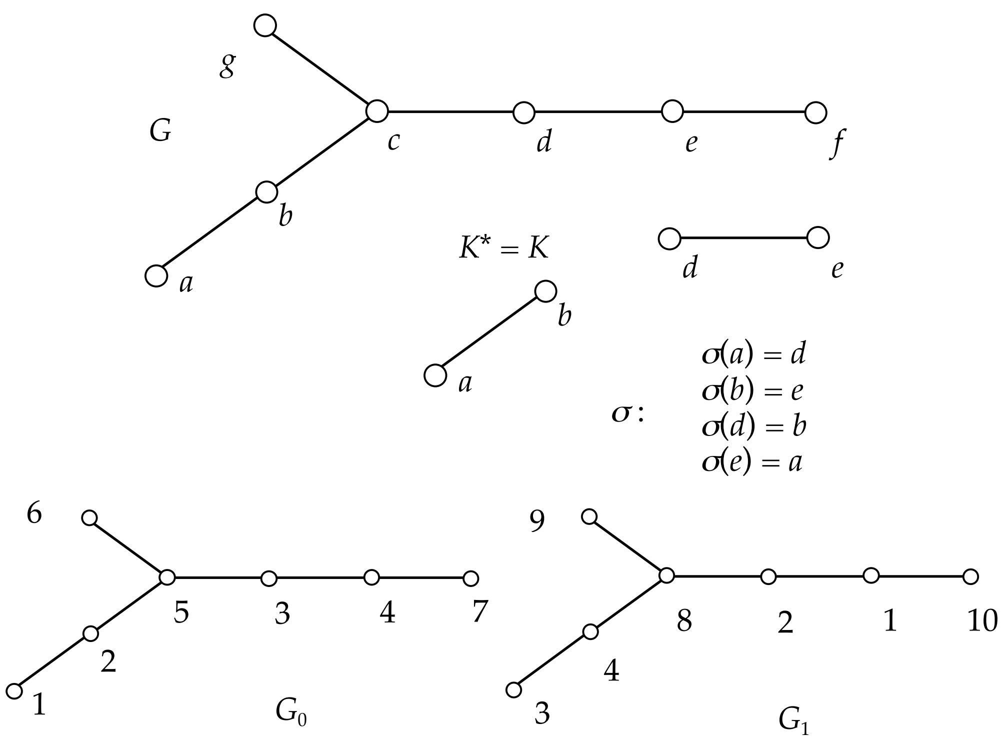

Figure 2. Linked pair construction for H.

To see more of the richness of the linked pair construction, let us examine two relatives of the example constructed in Fig. 2 and "deconstructed" in Fig. 1. For the first, let the ingredients G, K, K^* be the same as in Fig. 2, but for σ take the isomorphism $\sigma(a)=b, \sigma(b)=a, \sigma(d)=d, \sigma(e)=e$. Relabeling G like G_0 in Fig. 2 results in τ being $\tau(1)=2, \tau(2)=1, \tau(3)=3, \tau(4)=4, \tau(5)=8, \tau(6)=9, \tau(7)=10$. Then $G_1=\tau(G_0)$, and the new linked pair $H=G_0 \cup G_1$, are as shown in Fig. 3.

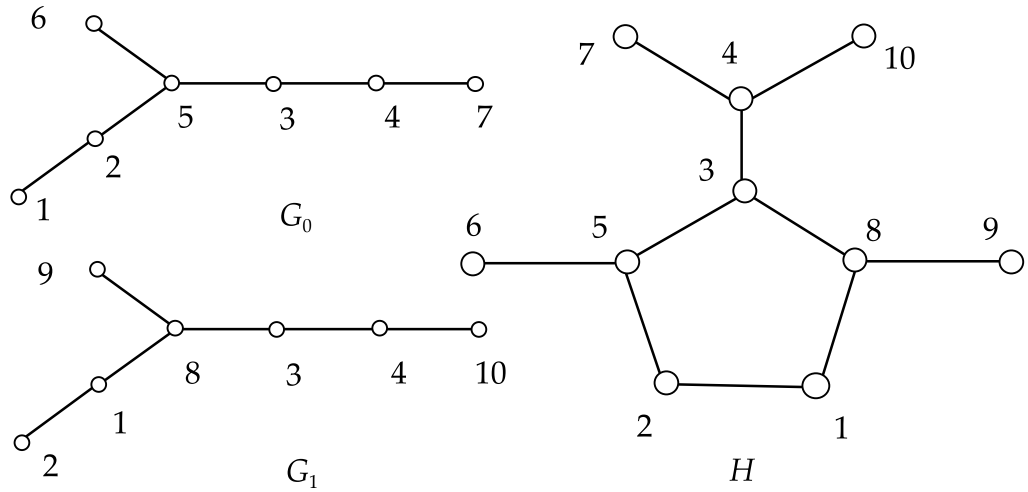

Figure 3. A linked pair related to Fig. 2.

For a further example, let G and K be the same as in Fig. 2, but now choose $K^*=G[b,c,e,f]$ and let σ be the isomorphism $\sigma(b)=a, \sigma(c)=b, \sigma(e)=e, \sigma(f)=d$. Relabeling G like G_0 in Fig. 2 results in τ being $\tau(1)=8, \tau(2)=1, \tau(3)=9, \tau(4)=4, \tau(5)=2, \tau(6)=10, \tau(7)=3$. The corresponding $G_1=\tau(G_0)$ is shown in Fig. 4, along with the new linked pair $H=G_0 \cup G_1$.

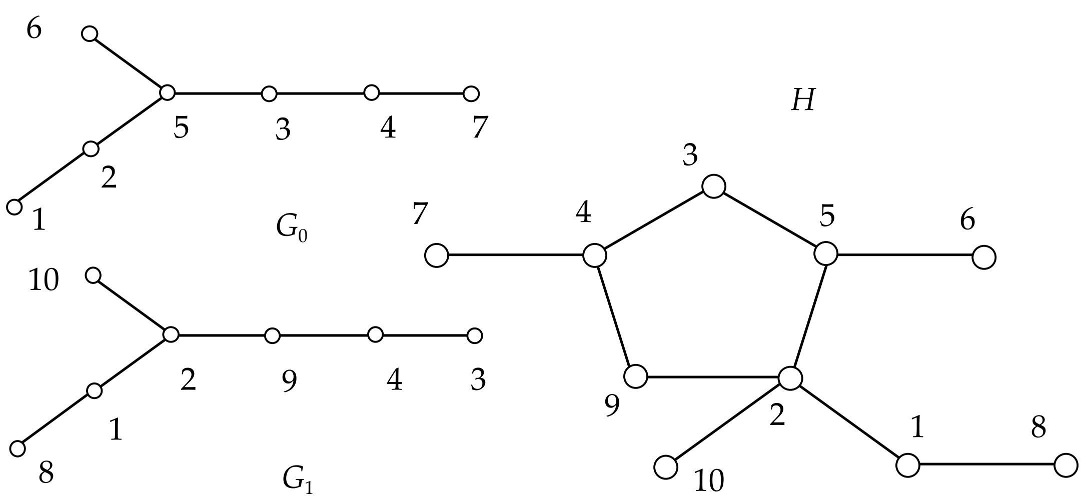

Figure 4. A third linked pair.

3. Linked chains

Iterating the linked pair construction produces a larger graph, comprising any desired number of isomorphic induced subgraphs which are linked together in a regular way. This larger graph is a *linked chain*.

As an example, let us iterate the linked pair construction shown in Fig. 2. For $G = G_0$ take the tree T with vertex set [1..7] and edge set {12, 25, 34, 35, 47, 56}, and for τ take $\tau(1) = 3$, $\tau(2) = 4, \tau(3) = 2, \tau(4) = 1$, $\tau(5) = 8$, $\tau(6) = 9$, $\tau(7) = 10$. Then the copy link is $G_1 = \tau(G_0)$. For each iteration, extend τ by adding an image vertex for each vertex not in the current domain of τ; take the last copy link as new initial link and act on it with the newly extended link isomorphism. Thus, for the first iteration τ is extended by adding $\tau(8) = 11$, $\tau(9) = 12$, $\tau(10) = 13$, producing the copy link $G_2 = \tau(G_1)$. For the second iteration, extend τ by $\tau(11) = 14$, $\tau(12) = 15$, $\tau(13) = 16$, producing $G_3 = \tau(G_2)$, and so on. For instance, the first four links are shown in Fig. 5. But it is not necessary to stop there: we could continue iteration and make as many links as we like.

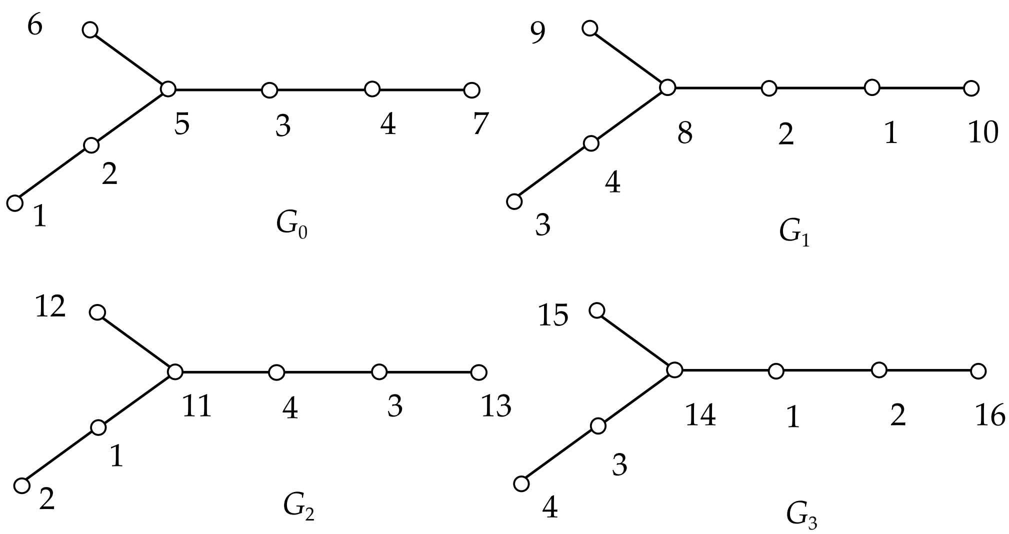

Figure 5. Four consecutive links of a linked chain.

The union of the first four links is the 4-linked chain $H = G_0 \cup G_1 \cup G_2 \cup G_3$. The regular structure of $H = H_4(G,\tau)$ is explicitly described by the equations

$$H[A \cup B \cup C \cup D] = H,$$

$$H[A] \cong H[B] \cong H[C] \cong H[D] \cong G, \quad H[A \cap B] \cong H[B \cap C] \cong H[C \cap D] \cong K,$$

where G is the initial link and K is the kernel of the linked pair construction. In our example the last chain of isomorphisms is actually a chain of three equalities, because we chose $K^* = K$ in the ingredients of the construction. The equations imply that A, B, C, D are subsets of the vertices of H, such that

$$|A| = |B| = |C| = |D| = g, \quad |A \cap B| = |B \cap C| = |C \cap D| = k.$$

We have $A = [1..7]$, $B = [1..4] \cup [8..10]$, $C = [1..4] \cup [11..13]$ and $D = [1..4] \cup [14..16]$ for our example, so $g = 7$ and $k = 4$; also $A \cap B = B \cap C = C \cap D = [1..4]$.

The behaviour of the link isomorphism τ is central to the structure of a linked chain, so we clarify the behaviour of τ with a visual representation. With the general labeling conventions used in Section 2, the linked pair $H = G_0 \cup G_1$ is labeled by $[1..(2g-k)]$, where G is labeled with $[1..g]$ and K is labeled with $[1..k]$. Then the τ-*digraph* is the directed graph $D(\tau)$ with vertex set $[1..(2g-k)]$ and with a directed edge $i \to \tau(i)$ for each $i \in [1..g]$. Since τ is one-to-one and has range $[1..k] \cup [(g+1)..(2g-k)]$, every vertex in $D(\tau)$ has outdegree at most 1 and indegree at most 1, and every vertex has positive total degree. Therefore every component of $D(\tau)$ has size at least 1, and is either a directed path or a directed cycle. The components of $D(\tau)$ are the *trajectories* of τ: a directed path is an *open* trajectory, and a directed cycle is a *closed* trajectory. There are $g - k > 0$ vertices of G which have indegree 0 in $D(\tau)$: each is the initial vertex of an open trajectory so every linked pair construction has at least one open trajectory. On the other hand, it is possible for a linked pair construction to have no closed trajectory. Note that an open trajectory of order μ comprises $\mu - 1$ vertices of $G = G_0$ and one terminal vertex which is in G_1 but not G_0; on the other hand, all μ vertices of a closed trajectory of order μ belong to $K^* \cap K \subset G = G_0$.

A single iteration of a linked pair construction simply involves extending its link isomorphism by adjoining to each open trajectory one new directed edge from the current terminal vertex to a new terminal vertex. Multiple iterations simply repeat this procedure: writing $D_0(\tau) = D(\tau)$ for the τ-digraph of the linked pair, let $D_n(\tau)$ denote the extended digraph after n further iterations of the linked pair construction. For example, Fig. 6 shows $D_0(\tau)$ and $D_2(\tau)$ for the linked pair construction in Fig. 2, representing respectively the first two links and all four links shown in Fig. 5.

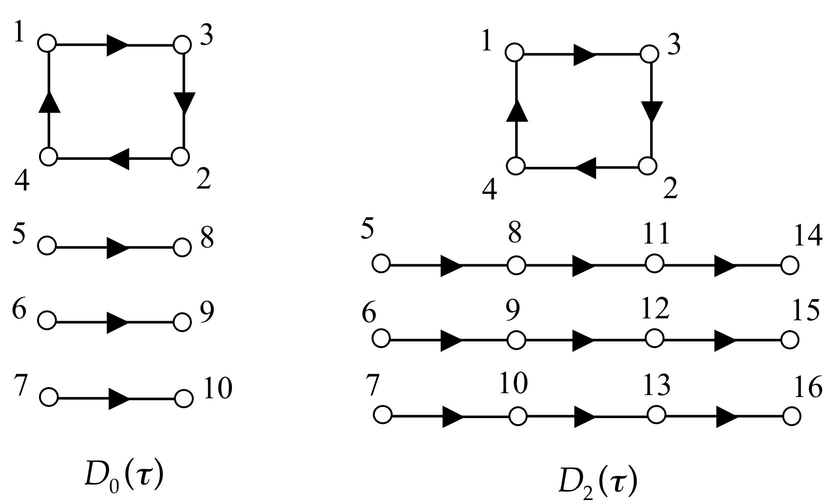

Figure 6. The τ-digraphs for Figs. 2 and 5.

How should we draw the 4-linked chain $H = H_4(G,\tau) = G_0 \cup G_1 \cup G_2 \cup G_3$ comprising the links in Fig. 5? There is no "best" drawing of *H*, but surely some drawings are better than others, insofar as they more clearly show the structural regularity of *H*. The iterated τ-digraph $D_2(\tau)$ reveals the essential structure of $H_4(G,\tau)$, so it is desirable to apply $D_2(\tau)$ to construct a drawing of $H_4(G,\tau)$. A simple method would be to take the arrangement of 16 vertices used for $D_2(\tau)$ in Fig. 6, omit all the directed edges, and insert the edges of the four links shown in Fig. 5. Another method is to arrange the vertices of each trajectory in $D_2(\tau)$ in a circle to reflect the rotational symmetry of the closed trajectory (Fig. 7).

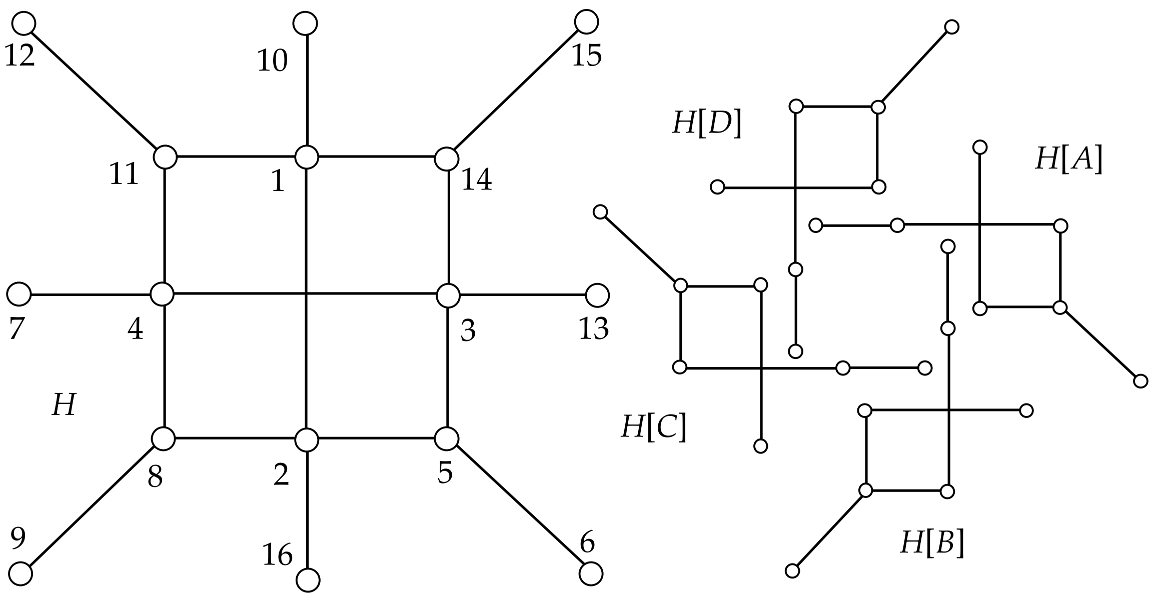

Figure 7. The 4-linked chain $H_4(G,\tau)$ with links in Fig. 5.

The τ-digraph $D(\tau)$ has another important use: the trajectories of τ show us how to create an *intrinsic labeling* of the vertices of the relevant linked pair, such that any associated linked chain has a natural extension of this labeling. Begin with the initial link *G* labeled by [1..*g*] and the kernel *K* labeled by [1..*k*]. Assign a linear ordering to the trajectories of τ corresponding to the linear ordering of their least members. For instance, the τ-digraph $D(\tau)$ in Fig. 6, corresponding to the linked pair in Fig. 2, has its trajectories in the linear order

$$(1,3,2,4) < \langle 5,8 \rangle < \langle 6,9 \rangle < \langle 7,10 \rangle .$$

Note that $\langle ... \rangle$ indicates an open trajectory (directed path), and $(...)$ indicates a closed trajectory (directed cycle). The new intrinsic labeling of any vertex *v* has the form (*i*, *j*), where the *primary index i* indicates the trajectory which contains *v*, and the *secondary index j* indicates the place of *v* in the sequence of all vertices comprising its trajectory, with the convention that the first vertex of a directed cycle is its least member in the prior labeling. If the *i*th trajectory has order μ, sequentially its vertices have intrinsic labels (*i*, *j*) with $0 \le j \le \mu - 1$, where $j \in \mathbf{Z}$ if the trajectory is open, and $j \in \mathbf{Z}_m$ if the trajectory is closed ($m = \mu$ is the *modulus* of the trajectory). For instance, the trajectories of $D(\tau)$ in Fig. 6 determine the following intrinsic vertex labeling for the linked pair in Fig. 2:

$$1 \leftrightarrow (1,0),\ 3 \leftrightarrow (1,1),\ 2 \leftrightarrow (1,2),\ 4 \leftrightarrow (1,3) \bmod 4;$$

$$5 \leftrightarrow (2,0),\ 8 \leftrightarrow (2,1); \quad 6 \leftrightarrow (3,0),\ 9 \leftrightarrow (3,1); \quad 7 \leftrightarrow (4,0),\ 10 \leftrightarrow (4,1).$$

In the first line all secondary indices are residues modulo 4 (the modulus of the closed trajectory is 4); in the second line all secondary indices are simply integers.

Given the intrinsic labeling of any linked pair, we can now neatly specify every linked chain which it generates. The key is the simplicity of the action of τ on the intrinsic labels of the vertices:

$$\tau(i,j) = (i,j+1),$$

where the second index increments by integer addition or by addition modulo m, depending on whether the ith trajectory is a directed path, or a directed cycle of order $\mu = m$. Of course, τ is a graph isomorphism, and its action on the edges is induced by its action on the vertices. So, if the initial link is $G_0 = G$, then the rth copy link is $G_r = \tau^r(G)$, and the n-linked chain is

$$H_n(G,\tau) = \bigcup\nolimits_{0 \le r < n} G_r = \bigcup\nolimits_{0 \le r < n} \tau^r(G),$$

where $\tau^r(i,j) = (i,j+r)$ for every integer $r \ge 0$. If V is the vertex set of the initial link $G_0 = G$, then $\tau^r(V)$ is the vertex set of $G_r = \tau^r(G)$. To illustrate, Fig. 8 shows the intrinsic labeling of the 4-linked chain in Fig. 7.

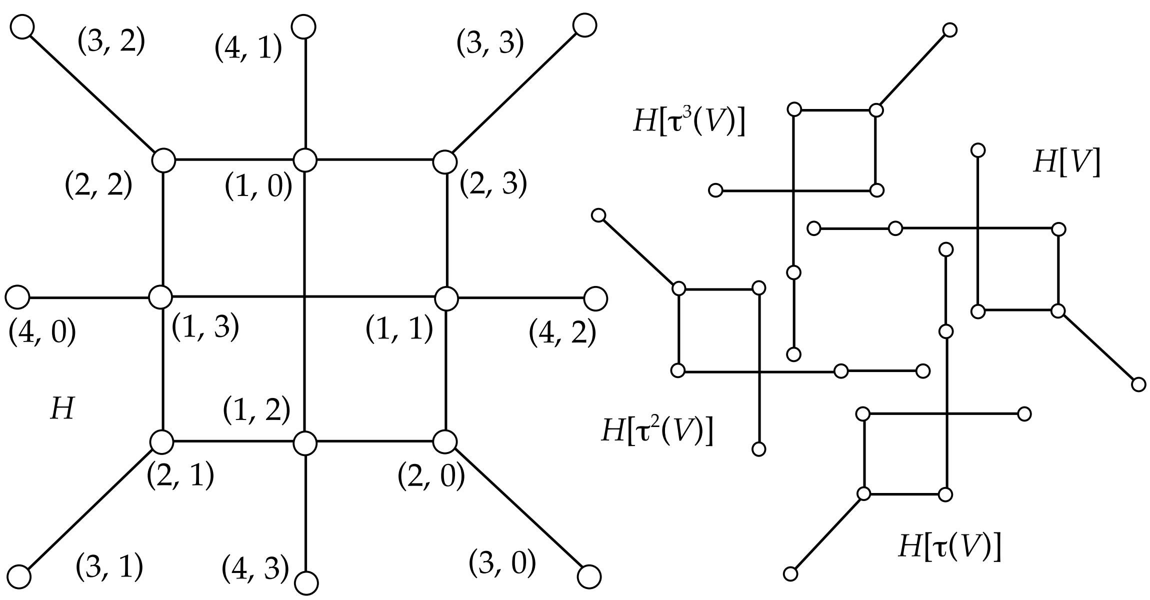

Figure 8. The intrinsic labeling of the 4-linked chain $H_4(G,\tau)$ in Fig. 7.

Now that we have seen how the τ-digraph $D(\tau)$ reveals the structure of a linked pair and its related linked chains, we can briefly examine the linked pairs in Figs. 3 and 4 to compare their structures with that of the linked pair in Fig. 2. For Fig. 3, the linear ordering of the trajectories in $D(\tau)$ is

$$\langle 1,2\rangle < \langle 3\rangle < \langle 4\rangle < \langle 5,8\rangle < \langle 6,9\rangle < \langle 7,10\rangle,$$

so the intrinsic labels for the vertices of the linked pair in Fig. 3 are

$$1 \leftrightarrow (1,0),\ 2 \leftrightarrow (1,1) \bmod 2; \quad 3 \leftrightarrow (2,0) \bmod 1; \quad 4 \leftrightarrow (3,0) \bmod 1;$$

$$5 \leftrightarrow (4,0),\ 8 \leftrightarrow (4,1); \quad 6 \leftrightarrow (5,0),\ 9 \leftrightarrow (5,1); \quad 7 \leftrightarrow (6,0),\ 10 \leftrightarrow (6,1).$$

For Fig. 4, the linear ordering of the trajectories in $D(\tau)$ is

$$\langle 5,2,1,8\rangle < \langle 7,3,9\rangle < \langle 4\rangle < \langle 6,10\rangle,$$

so the intrinsic labels for the vertices of the linked pair in Fig. 4 are

$$5 \leftrightarrow (1,0),\ 2 \leftrightarrow (1,1),\ 1 \leftrightarrow (1,2),\ 8 \leftrightarrow (1,3);$$

$$7 \leftrightarrow (2,0),\ 3 \leftrightarrow (2,1),\ 9 \leftrightarrow (2,2); \quad 4 \leftrightarrow (3,0) \bmod 1; \quad 6 \leftrightarrow (4,0),\ 10 \leftrightarrow (4,1).$$

The interested reader is invited to use this data to create drawings of the two 4-linked chains related to the one in Fig. 8.

4. Free linked chains

We have discussed n-linked chains for any positive integer n, but it is also natural to consider linked chains with infinitely many links. For a given linked pair with τ-digraph $D(\tau)$, we write $D_{+\infty}(\tau)$ to denote the extended digraph in which each open trajectory has been extended forward endlessly. The union of all the corresponding links is the *forward linked chain*,

$$H_{+\infty}(G,\tau) = \bigcup_{r\geq 0} G_r = \bigcup_{r\geq 0} \tau^r(G).$$

The mapping τ is a contractive isomorphism on $H_{+\infty}(G,\tau)$, so $H_{+\infty}(G,\tau)$ is isomorphic to an induced proper subgraph of itself. Indeed, for every $n \geq 1$,

$$H_{+\infty}(G,\tau) \cong \tau^n(H_{+\infty}(G,\tau)) = \bigcup_{r\geq n} \tau^r(G) = H_{+\infty}(G,\tau) - H_{n-1}(G,\tau).$$

It is also natural to specify $\tau^{-1}(i,j) = (i,j-1)$ for each intrinsic label (i,j), and to form the infinite digraph $D_{-\infty}(\tau)$ by modifying $D(\tau)$ so that each open trajectory extends backward endlessly. It is consistent with our earlier notation to write $G_{-r} = (\tau^{-1})^r(G) = \tau^{-r}(G)$ for each integer $r \geq 0$. The union of all these links is the *backward linked chain*,

$$H_{-\infty}(G,\tau) = \bigcup_{r\geq 0} G_{-r} = \bigcup_{r\geq 0} \tau^{-r}(G).$$

Again, the mapping τ^{-1} is a contractive isomorphism on $H_{-\infty}(G,\tau)$, so $H_{-\infty}(G,\tau)$ is isomorphic to an induced proper subgraph of itself. Indeed, for every $n \geq 1$,

$$H_{-\infty}(G,\tau) \cong \tau^{-n}(H_{-\infty}(G,\tau)) = \bigcup_{r\geq n} \tau^{-r}(G) = H_{-\infty}(G,\tau) - H_{1-n}(G,\tau)$$

However, more important than either of the above infinite linked chains is the *free linked chain*, the union of all forward and backward links,

$$H_{\pm\infty}(G,\tau) = \bigcup_{r\in \mathbf{Z}} G_r = \bigcup_{r\in \mathbf{Z}} \tau^r(G).$$

The mapping τ is a nontrivial automorphism of $H_{\pm\infty}(G,\tau)$, so for every $n \in \mathbf{Z}$,

$$\tau^n(H_{\pm\infty}(G,\tau)) = H_{\pm\infty}(G,\tau).$$

Thus free linked chains are highly symmetric: they have infinite automorphism groups. In the next section we use free linked chains to construct infinite families of new graphs with nontrivial automorphism groups. Free linked chains are particularly important because of their role in this construction.

We close this section by noting that the infinite linked chains described here correspond to the infinite extensions $D_{+\infty}(\tau)$, $D_{-\infty}(\tau)$ and $D_{\pm\infty}(\tau)$ of the τ-digraph $D(\tau)$ of the underlying linked pair construction. Fig. 9 shows the free τ-digraph $D_{\pm\infty}(\tau)$ for the linked pair introduced in Fig. 2.

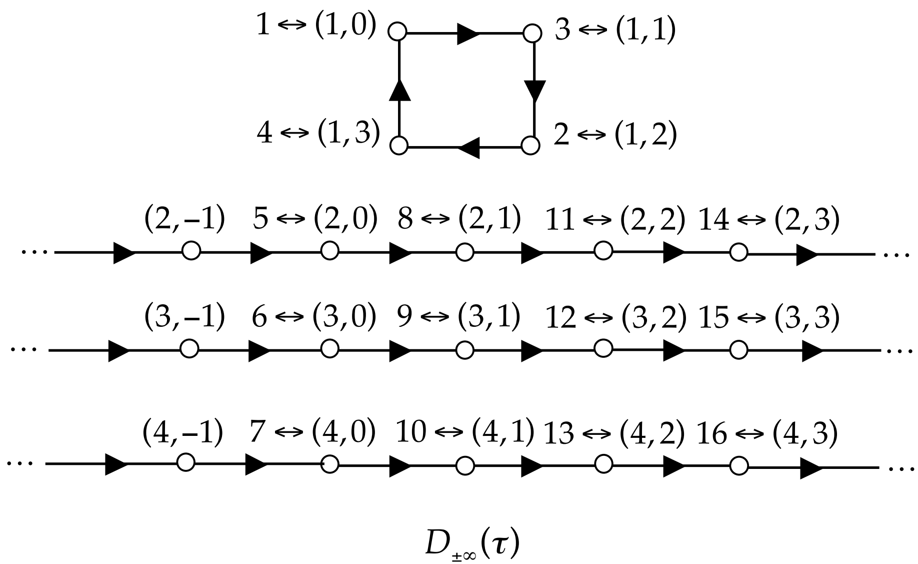

Figure 9. The free τ-digraph for the linked pair in Fig. 2.

5. Linked cycles

A *linked cycle* is a special kind of homomorphic image of a free linked chain, produced by collapsing each open trajectory of τ into a closed trajectory, so that the resulting graph is a cyclic sequence of isomorphic induced subgraphs (links) in which consecutive pairs of links all have isomorphic intersections.

How is this collapsing of open trajectories achieved? If the ith trajectory in $D(\tau)$ is an open trajectory of order μ, to collapse the corresponding trajectory of the free linked chain $H_{\pm\infty}(G,\tau)$ we choose a positive integer $m \geq \mu$ and make the identification $(i,a) = (i,b)$ whenever $a \equiv b \pmod{m}$. Formally, this replaces $\mathbf{Z}$ by $\mathbf{Z}_m$ as the secondary index set for vertices in the ith trajectory. It is natural to call m the *modulus* of the collapsed trajectory, matching the terminology already in place for closed trajectories. Collapse all open trajectories in this way; in the resulting graph two vertices are adjacent whenever they are equivalent to two vertices which are adjacent in the free linked chain. If there are t trajectories, and $(m_1, m_2, ..., m_t)$ is the list of moduli, the resulting graph may be denoted by

$$H'(G,\tau) = H_{\pm\infty}(G,\tau)/(m_1, m_2, ..., m_t).$$

Also τ', defined by $\tau'(i,j) = (i, j+1) \bmod m_i$ for $1 \leq i \leq t$, is an automorphism.

In order for $H'(G,\tau)$ to qualify as a *linked cycle constructed from G and* τ we require it to be appropriately composed of induced copies of *G*, specifically:

(1) *G* is the subgraph induced by the vertex set $V = \{(i,j) : 0 \le j < m_i, 1 \le i \le t\}$;

(2) *G* is isomorphic to each of the subgraphs G'_r such that $G'_0 = G$ and $G'_{r+1} = \tau'(G'_r)$ for all $r \ge 0$;

(3) $H'(G,\tau)$ is the union of finitely many of the subgraphs G'_r;

(4) the intersections $G'_r \cap G'_{r+1}$ are isomorphic for all $r \ge 0$.

Hence, in general there are subtle further constraints on the choice of moduli for collapsing the open trajectories of $H_{\pm\infty}(G,\tau)$, so that nonadjacent vertices of *G* do not become adjacent as a result of the collapsing operation.

To avoid improper adjacencies *within* a collapsed trajectory of order μ, it suffices for its modulus to satisfy $m \ge 2\mu - 2$. Improper adjacencies can only occur *between* two trajectories if at least one of them is a collapsed trajectory. If both trajectories are collapsed trajectories, of orders μ and μ', it suffices for their moduli m and m' to satisfy $\gcd\{m,m'\} \ge \max\{\mu,\mu'\}$. If one is a closed trajectory of order μ and modulus $m = \mu$, and the other is a collapsed trajectory of order μ' and modulus m', there can be no improper adjacency if $m \mid m'$ and $m' \ge \mu'$.

We can now specify a class of linked cycles constructed from *G* and τ. Let m_0 be the least common multiple of the moduli of the closed trajectories ($m_0 = 1$ if there are no closed trajectories, by the usual convention for product over an empty set), let μ_0 be the maximum of the moduli of the open trajectories, and let *c* be a positive integer such that $cm_0 \ge 2\mu_0 - 2$. If the *i*th trajectory is open, choose its modulus for $H'(G,\tau)$ to be $m_i = cm_0 n_i$, where n_i is any positive integer. Any such choice of $(m_1, m_2, ..., m_t)$ satisfies all the sufficient conditions for $H'(G,\tau)$ to be a linked cycle constructed from *G* and τ.

To illustrate, for any choice of positive integers a_2, a_3, a_4, the list of moduli $(m_1, m_2, ..., m_t) = (4, 4a_2, 4a_3, 4a_4)$ applied to the free linked chain generated by the linked pair in Fig. 2 yields an *n*–linked cycle with $n = 4 \cdot \mathrm{lcm}\{a_2, a_3, a_4\}$. Indeed, when $(m_1, m_2, ..., m_t) = (4,4,4,4)$, the 4-linked cycle is precisely the graph in Fig. 7 but with $\mathbf{Z}_4$ as the second index set of all four trajectories. Again, for any positive integers a_4, a_5, a_6, the list of moduli $(m_1, m_2, ..., m_t) = (2,1,1,2a_4,2a_5,2a_6)$ applied to the free linked chain generated by the linked pair in Fig. 3 yields an *n*–linked cycle with $n = 2 \cdot \mathrm{lcm}\{a_4, a_5, a_6\}$. Indeed, when $(m_1, m_2, ..., m_t) = (2,1,1,2,2,2)$, the 2-linked cycle is precisely the graph in Fig. 3 but with $\mathbf{Z}_1$ as the second index set of the second and third trajectories, and $\mathbf{Z}_2$ as the second index set of the other four trajectories. Again, applying $(m_1, m_2, ..., m_t) = (6a_1, 6a_2, 1, 6a_4)$ to collapse the free linked chain generated by the linked pair in Fig. 4 yields an *n*–linked cycle with $n = 6 \cdot \mathrm{lcm}\{a_1, a_2, a_4\}$.

However, note that the lists of moduli constructed above are in general not the only possibilities for collapsing a free linked chain into a linked cycle. For example, $(m_1, m_2, ..., m_t) = (4,4,1,4)$ applied to the free linked chain generated by the linked pair in Fig. 4 yields a 4–linked cycle.

To close this section, observe that if a linked pair has more than one open trajectory, its free linked chain can be *partially* collapsed, by choosing moduli for a proper subset of the open trajectories, consistent with the constraints already noted, while leaving the other open trajectories unchanged. The resulting infinite graph is a *constrained linked chain*, and the corresponding τ' is an automorphism. (If the *i*th trajectory is open, assigning $m_i = 0$ to be its modulus is notationally consistent and leaves the trajectory uncollapsed; we can adapt the earlier class of linked cycle solutions to include constrained linked chains by allowing a_i to be a nonnegative integer rather than requiring it to be a strictly positive integer.)

6. Symmetric graphs as linked cycles

We began this discussion with a particular graph (Fig. 1) and showed how it can be "factorized" into two isomorphic subgraphs with nonempty intersection. This is an example of the "analytic" viewpoint, where something is taken apart so that its pieces can be examined. Its complement is the "synthetic" viewpoint, where separate pieces are assembled into a whole, and we are interested in the final product, or the range of possible final products. Beginning with the pieces in Fig. 2, which fit together to form our first graph, most of our discussion has been from the synthetic viewpoint: we have seen how to build linked pairs, then finite linked chains, then free linked chains, then linked cycles and constrained linked chains. In this final section we return to the analytic viewpoint, and see that the presence of symmetry in a graph typically allows us to "factorize" it as a linked cycle.

Let α be any automorphism of a given finite graph *F*. The α-*digraph* is the directed graph $D(\alpha)$ on the vertices of *F*, such that $v \to \alpha(v)$ is a directed edge for each vertex *v* of *F*. The components of $D(\alpha)$ are the *trajectories* of α. Since α is a bijection, every vertex has outdegree 1 and indegree 1, so each trajectory is closed (a directed cycle). If α is the identity map, then all trajectories have order 1; in every other case at least one trajectory has order greater than 1. A trajectory of α is *primal* if its vertex set induces a subgraph K_1 or K_2 in *F*; all other trajectories are *derived*. This terminology can be extended to α itself: we say α is *primal* if all its trajectories are primal, and is *derived* if at least one of its trajectories is derived.

Let *F* be a finite graph with a derived automorphism α. We can always find a proper induced subgraph *G* and a link isomorphism τ such that

$$F = H'(G,\tau) = H_{\pm\infty}(G,\tau)/(m_1, m_2, \ldots, m_t) \quad \text{and} \quad \alpha = \tau' = \tau/(m_1, m_2, \ldots, m_t).$$

In other words, *F is a linked cycle with links isomorphic to a proper induced subgraph G, and its link isomorphism is* $\tau' = \alpha$. Thus, every finite graph with a derived automorphism can be "cyclically factorized" into isomorphic links (proper induced subgraphs) in such a way that consecutive pairs intersect isomorphically.

To illustrate, the graph *F* in Fig. 10 has an automorphism α of order 2. In this instance α has two trajectories of order 1 and two of order 2, whence the intrinsic labeling of vertices in Fig. 10. The first three trajectories are primal; the fourth is derived, since its vertices induce a subgraph $2K_1$ in *F*. Take a map τ with free τ-digraph $D_{\pm\infty}(\tau)$ comprising three closed trajectories identical with those of α and one infinite trajectory, so that $\alpha = \tau' = \tau/(1,1,2,2)$. Then τ itself has closed trajectories identical with those of $D_{\pm\infty}(\tau)$, and an open trajectory of order $\mu \geq 2$. The infinite trajectory of $D_{\pm\infty}(\tau)$ requires a modulus $m = 2$; also $m \geq \mu$ is necessary, so we have $\mu = 2$. The set $V = \{(1,0), (2,0), (3,0), (3,1), (4,0)\}$, comprising the vertices of the closed trajectories and the nonterminal vertex of the open trajectory of τ, is the vertex set of the initial link $G = F[V]$ of the free linked chain $H_{\pm\infty}(G,\tau)$, and $F = H'(G,\tau) = H_{\pm\infty}(G,\tau)/(1,1,2,2)$. Of course τ is the restriction of α to G. Thus F is a 2-linked cycle with initial link $G = P_5$ (Fig. 10). The kernel of the underlying linked pair is

$$K = G \cap \alpha(G) = F[V \cap \alpha(V)] = F[(1,0), (2,0), (3,0), (3,1)] = 2K_2,$$

and the prekernel is $K^* = \alpha^{-1}(K) = K$; the shift σ is the restriction of α to K^*. But note that adding an edge to F between (4,0) and (4,1) would make α primal, and the modified graph would no longer be a linked cycle.

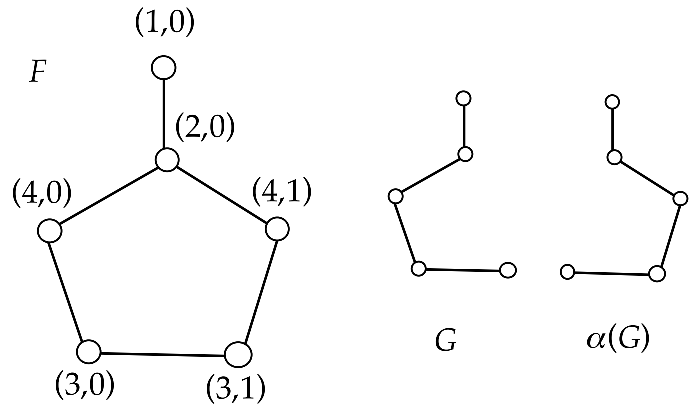

Figure 10. A symmetric graph shown as a linked cycle.

As a final example, consider P, the Petersen graph[1]. This fascinating graph has many delightful features. The intrinsic labeling of P in Fig. 11 corresponds to the trajectories of an automorphism α of order 5. Both trajectories have order 5, and both induce the subgraph C_5, so both are derived. Take a map τ with free τ-digraph $D_{\pm\infty}(\tau)$ comprising two infinite trajectories, so that $\alpha = \tau' = \tau/(5,5)$. Then τ itself has two open trajectories. The order of each must satisfy $\mu \leq 5$, the bound imposed by the modulus $m = 5$ applied to the trajectories of $D_{\pm\infty}(\tau)$ to give $\tau' = \alpha$.

Given that the ith trajectory is open, define the *longest edge length* λ of the ith trajectory to be the largest integer λ such that (i, λ) is adjacent to $(i, 0)$ in P. Then the order of the ith trajectory must satisfy $\mu > \lambda$, so the order of the first trajectory must satisfy $2 \leq \mu \leq 5$, and that of the second trajectory must satisfy $3 \leq \mu \leq 5$. The smallest possible solutions minimize the order of the initial link, so take $\mu = 2$ for the first trajectory and $\mu = 3$ for the second. The set comprising the nonterminal vertices of the open trajectories of τ is $V = \{(1,0), (1,1), (2,0), (2,1), (2,2)\}$, so the induced subgraph $G = P[V]$ is the initial link of a free linked chain $H_{\pm\infty}(G,\tau)$ such

[1] The Petersen graph was used as the logo for the Severino V. Gervacio Conference on Graph Theory and Combinatorics, De La Salle University, April 24-25, 2009.

that $P = H'(G,\tau) = H_{\pm\infty}(G,\tau)/(5,5)$, and τ is the restriction of α to G. Here $G = P_5$ generates the Petersen graph P as a 5-linked cycle (Fig. 11). The kernel of the underlying linked pair is

$$K = G \cap \alpha(G) = P[V \cap \alpha(V)] = P[(1,1),(2,1),(2,2)] = K_1 \cup K_2,$$

the prekernel is $K^* = \alpha^{-1}(K) = P[(1,0),(2,0),(2,1)]$, and the shift is $\sigma = \alpha \mid K^*$.

Note that there are 12 solutions for the orders of the τ trajectories, and each gives rise to a "factorization" of P as a 5-linked cycle. The case in Fig. 11 is for the pair of orders (2, 3). The pair (3, 3) has $V = \{(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$, so in this case the initial link is $G = P[V] = F$, where F happens to be a 5-cycle with a pendant vertex, the graph in Fig. 10. The pair (2, 4) also has $G = F$, but on a different vertex set V. The pair (5, 5) is the largest solution, and in this case G is a pair of 5-cycles with a single edge in common. In all 12 solutions the initial link G contains the initial link P_5 of the smallest solution; moreover, the kernel and prekernel contain the corresponding subgraphs of the smallest solution.

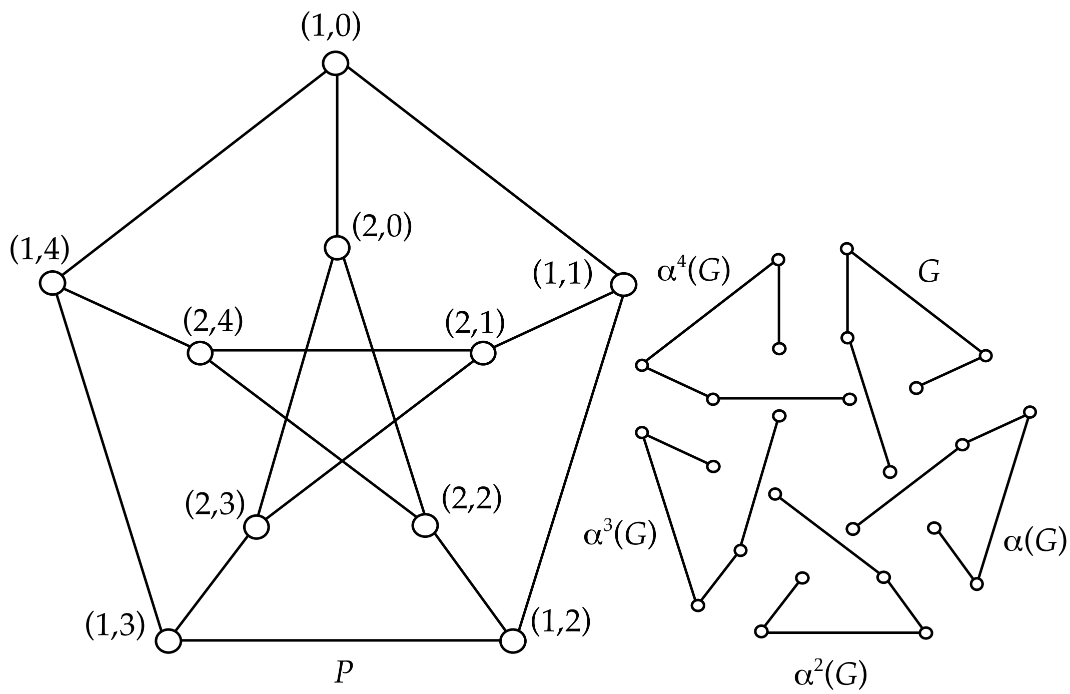

Figure 11. The Petersen graph as a linked cycle.

As another application of this analytic approach, the interested reader is invited to calculate the "factorization" of P resulting from an automorphism of order 2, having two derived trajectories.

This introduction is the first paper to appear on graphs as linked structures. We anticipate the forthcoming publication of a more formal account [**1**].

Reference

[**1**] Roger Eggleton, Peter Adams, and James MacDougall, *Graphs which are linked structures*, to appear.