Corrections to Steinbach's Posets of Graphs (Orders 5, 6, 7)

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Abstract. Steinbach's useful tabulations of the posets of graphs of orders 5, 6 and 7 (in his *Field Guide to Simple Graphs*) are marred by a sparse scattering of errors. We list all corrections needed, and for convenience provide the full, corrected data at

`http://www.maths.uq.edu.au/~pa/research/steinbach.html`.

1 Introduction

How many unlabelled simple graphs have degree sequence 1222333, and what do they all look like? How many unlabelled simple graphs with 7 vertices and 10 edges (*order* 7, *size* 10) are connected, and which among them are hamiltonian?

Peter Steinbach's *Field Guide to Simple Graphs* [**2,3**] is a very handy tool that enables the practitioner of graph theory to answer such questions quickly and conveniently. However, in the course of recent work we became aware of a number of errors in Steinbach's subgraph tabulations. Subsequently we independently recalculated the corresponding tables, identified all the discrepancies, and verified that each discrepancy was a genuine correction. Our purpose in the present note is to report these corrections so that all who wish to make full use of Steinbach's tables can confidently do so. In a private communication, Peter Steinbach has indicated to us that the corrections will be incorporated in future printings of the *Field Guide*.

We note that Read and Wilson's *Atlas* [**1**] is also handy for answering questions like those in our opening paragraph. However, Steinbach's organisation and numbering system make [**2,3**] more convenient for some applications, especially those in which subgraphs and complementation are relevant.

2 Posets of Graphs (Orders $5, 6, 7$)

To introduce the corrections in their proper context, we need some notation and terminology. Let G and H be any unlabelled simple graphs of order n. If adding a suitable finite set E of edges to G produces a graph $G + E$ which is isomorphic to H, then H is an *extension* (spanning supergraph) of G, or equally, G is a *reduction* (spanning subgraph) of H, and we write $G \leq H$. If $|E| = 1$, then H is a 1-*extension* of G, and G is a 1-*reduction* of H. If $G \leq H$, the complements satisfy $H^c \leq G^c$. Let $\mathcal{G}(n)$ be the partially ordered set of all unlabelled simple graphs of order n, with this partial ordering. The poset $\mathcal{G}(n)$ has the complete graph K_n as maximum element, and its complement the empty graph K_n^c as minimum element. The mth *level set* $\mathcal{G}(n, m)$, comprising all unlabelled simple graphs of order n and size m, is a maximal independent subset in $\mathcal{G}(n)$. Every maximal ascending chain in $\mathcal{G}(n)$ begins with K_n^c and ends with K_n and contains exactly one graph from each level set.

Steinbach specifies the posets $\mathcal{G}(n)$, $n \leq 7$ on pp. 90–107 of [**2,3**]. Below we report corrections for $\mathcal{G}(5)$, $\mathcal{G}(6)$ and $\mathcal{G}(7)$. Steinbach assigns numbers to the graphs in each of these posets so that the 1-reductions of any graph G have smaller numbers than G, and the 1-extensions have larger numbers. Moreover, in $\mathcal{G}(6)$ any graph and its complement have numbers x and x^c satisfying $x + x^c = 157$ (since $|\mathcal{G}(6)| = 156$); in $\mathcal{G}(7)$ the corresponding identity is $x + x^c = 1045$. In $\mathcal{G}(5)$ most complementary pairs satisfy $x + x^c = 35$, but here the situation is complicated by the presence of two self-complementary graphs (numbered 17 and 19); the graphs numbered 16, 17 and 18 satisfy $x + x^c = 34$. Steinbach specifies $\mathcal{G}(5)$ and $\mathcal{G}(6)$ by listing all 1-reductions and 1-extensions of each graph. For $\mathcal{G}(7)$, the corresponding lists are given explicitly only for graphs with numbers $x \leq 522$, thereby saving 11 pages; the lists for $x \geq 523$ can be readily deduced by using complementation.

The errors in Steinbach's tables occur in the lists of 1-reductions and/or 1-extensions of certain graphs. For each such graph we specify the corrections needed simply by giving the correct list of all 1-reductions and 1-extensions. The reader will easily be able to apply these corrections to any copy of [**2**,**3**].

A few errors present in [**2**] are corrected in [**3**]. For example, graph 6 has graph 10 as a 1-extension in $\mathcal{G}(6)$. This fact is omitted from the lists of 1-reductions and 1-extensions of both graph 6 and graph 10 on p. 94 of [**2**], but is corrected in [**3**]. Again, the graphs with numbers 513–532 had their numbers omitted from p. 89 of [**2**], but this is corrected in [**3**].

3 Corrections

The following corrections all apply to pp. 93–107 of [**2**]. With the exception of the lines for graphs 6 and 10 of $\mathcal{G}(6)$, every correction also applies to [**3**]. Table 1 gives corrections to $\mathcal{G}(5)$, Table 2 gives corrections to $\mathcal{G}(6)$ and Table 3 (at the end of this note) gives corrections to $\mathcal{G}(7)$.

Table 1: **Corrections to $\mathcal{G}(5)$**

18	**22**	27 30
15 16 18 19	**23**	27 28 29
16 20	**25**	28
15 18 19 20	**26**	28 29 30

Table 2: **Corrections to $\mathcal{G}(6)$**

3	**6**	10 12 13 16
5 6 7	**10**	19 20 21 22 28
35 40 41 44	**59**	99 100 101 102
36 46 49	**64**	79 89 90 97
36 45 46 50 51	**70**	79 83 84 85 88 89
38 42 46 51 52	**71**	80 85 89 91
38 46 47 48 49 50 52	**72**	80 81 83 86 87 89 90 96
72 75 77	**81**	103 106 109
55 60 61 64	**97**	108 116 122
56 59 60 69	**102**	121 122 123
129 135 136 137 138	**147**	150 151 152
141 144 145 147	**151**	154

4 Website Availability

As a public service, we have placed correct tables for $\mathcal{G}(5)$, $\mathcal{G}(6)$ and $\mathcal{G}(7)$ on the website

`http://www.maths.uq.edu.au/~pa/research/steinbach.html`.

These tables retain the numbering scheme used by Steinbach. They list the 1-reductions and 1-extensions of each graph of order 5, 6 or 7. To make the website relatively self-contained, we have also specified the Steinbach reference number, the degree sequence and the edge set of each graph of order 5 or 6, and of each graph of order 7 and size at most 10. (Complementation and the identity $x + x^c = 1045$ readily yield the corresponding information for any order 7 graph of size greater than 10.)

References

[**1**] Ronald C. Read and Robin J. Wilson, *An Atlas of Graphs*, Oxford University Press (1998).

[**2**] Peter Steinbach, *Field Guide to Simple Graphs*, second edition (1995), published by Design Lab, Albuquerque Technical-Vocational Institute, Albuquerque, NM.

[**3**] — *ibid.*, second *revised* edition (1999).

Table 3: **Corrections to $\mathcal{G}(7)$**

20 29 33	**52**	91 94 97 112 119 128
24 30	**55**	87 90 96 99 105 118 122 136
24 28 31 32	**56**	88 95 99 100 103 117 122 135
25 30 31 32	**59**	95 98 99 104 105 127 138 139 143
27 30 35 37	**66**	107 111 113 116 118 124 127
30 32 36 37 40	**75**	113 117 121 127 131 133 136 137 139 141 144
45 53 54 55 56 57 59	**99**	153 155 157 159 165 166 167 169 198 200 201 208 209 227
55 58 59	**105**	161 166 169 170 223 233 234
42 52 65 68 74	**112**	177 182 185 187 190 194 215
50 54 64 70 72	**115**	176 181 186 196 200 213 218
51 56 64 67 70 75	**117**	173 176 183 188 198 200 202 203 218 220
43 52 68 71 74	**119**	177 190 193 195 204 206 225
53 54 69 70 72 75	**121**	176 179 192 196 198 201 207 208 224 231
51 58 67 74 76	**126**	182 195 199 203 212 216 219 220 221 223 235
59 66 67 75	**127**	183 197 198 222 223 236
52 72 79	**128**	196 205 207 215 225 237
59 70 76 81	**138**	199 200 209 222 231 233 234 240
59 69 75 76	**139**	188 201 208 223 231 232 234 243
90 97 104 105	**170**	259 260 262 336 337 342
92 93 107 110 114 115 124	**181**	272 276 295 296 300 301 302 305 345
86 94 110 112 114 126	**182**	273 276 277 295 297 298 300 303 304 344
95 108 114 117 139	**188**	272 286 298 311 312 315 348
88 94 114 119 120 125 132 135	**193**	282 284 285 290 301 303 308 311 313 322 324 328 359
88 97 112 113 120 124 131 135	**194**	276 283 285 294 304 307 308 309 310 320 323 337 359
89 97 114 119 126 134	**195**	277 282 293 302 308 311 318 329 336 354
95 98 109 114 120 126 135 138	**199**	276 282 286 288 304 310 311 313 326 330 333 334 356

Table 3: **Corrections to** $\mathcal{G}(7)$ (continued)

95 99 113 118 121 122 124 139	**201**	272 283 287 294 306 312 314 316 323 325 327 334 361
96 100 117 122 126 136	**203**	273 286 293 306 315 316 317 331 332 358
94 112 124 128 131 142	**215**	300 309 314 320 321 323 344 349 351 359
134 137 141 142 145	**239**	351 352 354 357 365 366 368
157 159 163 165 166 167 170	**260**	381 383 387 388 434 441 442 443 455
152 153 172 173 175 176 181 188 201	**272**	391 393 398 405 406 408 409 412 413 414 421 467
153 155 178 179 187 192 194 201 208	**283**	393 401 403 413 415 420 424 425 428 438 454 498
162 184 189	**299**	404 423 461 470
164 173 182 183 194 199 212 220 222	**304**	410 412 419 420 422 436 437 466 467 476 477 480
162 203 221	**317**	407 431 449 470 491
165 191 192 199 208 219 227 230 231	**326**	425 429 432 442 444 454 455 477 484 496 509 517
166 200 203 209 210 218 219 227 233	**331**	416 429 442 445 449 451 456 481 490 492 506 510
199 219 226 235 240	**356**	476 483 487 488 510 515 517
193 194 215 222 225 230 236 240 241	**359**	476 477 483 484 493 495 498 499 503 512 514 518
216 217 222 223 232 234 242	**362**	468 473 480 494 497 508 511 512 513
207 224 225 237 238 239	**365**	482 501 502 503 505 522
255 257 260 261 262 264	**388**	568 569 581 582 613 635
245 246 266 267 272 273 274 276 283 286	**393**	617 624 632 636 637 639 652 653 654 666
248 283 287 294	**403**	545 578 583 626 627 654
257 272 276 278 286 298 305 310 315 334	**421**	603 607 613 618 621 628 629 632 644 652 664
260 279 293 302 305 306 325 336	**434**	589 597 599 601 610 613 629 663
299	**461**	646 670
300 301 304 305 324 326 345 355 359 360	**477**	588 589 590 591 592 593 601 603 616 621 657
334 341 356 361 363 370	**515**	524 527 533 534 547 607
290 320 359 360 372 373	**518**	530 535 537 548 555 588